Nonhomogeneous ODE

$L := a_n D^n + a_{n-1} D^{n-1} \cdots + a_0$

$D := \frac{d}{dx}$

Consider: $Ly = g(x)$ ①

Solution (superposition principle) $y = y_c + y_p$ where y_c is the general solution of

$Ly = 0$ ②

y_p: particular solution

Next lectures: Learn how to find y_p!

4.4. Methods of undetermined coefficients

+ General idea: guess the solution y_p from the form of $\boxed{g(x) \text{ and } y_c.}$

+ Applicable if $g(x)$ is a sum of terms of the following forms:

- $P(x)$: polynomial, i.e, $b_n x^n + b_{n-1} x^{n-1} \cdots + b_1 x + b_0$
- $P(x) e^{ax}$
- $P(x) e^{\alpha x} \cos(\beta x)$, $P(x) e^{\alpha x} \sin(\beta x)$

+ $y_p = x^m$ (linear sum of functions that are generated by repeated differentiations of $g(x)$)

m: depends on how $g(x)$ is related to y_c.

Consider $ay'' + by' + cy = g(x)$

Homogeneous eq: $ay'' + by' + cy = 0$

Char eq: $\underline{ar^2 + br + c = 0}$. (*)

Magic Box: $g(x)$ vs y_c

Case 1: $g(x) = P(x) e^{r_0 x}$, $P(x)$ polynomial of order k

$$y_p = x^m \underbrace{(A_k x^k + \dots + A_1 x + A_0) e^{r_0 x}}_{\text{sum of derivatives of } g(x)}$$

$m = 0$ if r_0 is not a root of (*)

$m = 1$ if r_0 is a simple root of (*)
($e^{r_0 x}$ is a part of y_c)

$m = 2$ if r_0 is a double root of (*)

Case 2: $g(x) = \begin{cases} P(x) e^{\alpha x} \cos(\beta x) \\ P(x) e^{\alpha x} \sin(\beta x) \end{cases}$ $P(x)$ poly of degree k

$$y_p = x^m \Big[\underbrace{(A_k x^k + \cdots + A_1 x + A_0) e^{\alpha x} \cos(\beta x) + (B_k x^k + \cdots + B_1 x + B_0) e^{\alpha x} \sin(\beta x)}_{\text{derivatives of } g(x)} \Big]$$

$m = 0$ if $\alpha + i\beta$ is not a root of (*)
$m = 1$ if $\alpha + i\beta$ is a root of (*)
(never have double complex roots)

<u>Ex</u>: $y'' - 2y' + y = 2e^x$, find y_p

$g(x) = 2e^x \Rightarrow P(x) = 2$ of degree 0
$r_0 = 1$

Char eq: $r^2 - 2r + 1 = 0$ has a double root $r = 1$

$\Rightarrow y_p = x^m (Ae^x)$, $m = 2$
$= x^2 A e^x$ (NOT the final answer)

Determine A:

$y_p' = 2Axe^x + Ax^2e^x$
$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$
$= Ax^2e^x + 4Axe^x + 2Ae^x$

$y_p'' - 2y_p' + y = 2e^x$

$\cancel{Ax^2e^x} + \cancel{4Axe^x} + 2Ae^x - \cancel{4Axe^x} - \cancel{2Ax^2e^x} + \cancel{x^2Ae^x} = 2e^x$

$2Ae^x = 2e^x \Rightarrow A = 1.$

<u>Ex</u>: $y'' - 2y' + 2y = 2e^x \cos x$

$P(x) = 2$ degree 0, $\alpha = 1$, $\beta = 1$

Char eq: $r^2 - 2r + 2 = 0$

has complex roots $r = 1 \pm i$.

Observe $\alpha + i\beta = 1 + i$ is a root of the char eq.

$y_p = x^m (A e^x \cos x + B e^x \sin x)$

$m = 1$.

$y_p = x (A e^x \cos x + B e^x \sin x)$.

Next step: Determine A, B (leave it for you)

Ex: $y'' - 4y' + 3y = x^2 + x - 1 + \sin x$

$g(x) = \underbrace{x^2 + x - 1}_{P(x)} + \underbrace{\sin x}_{\sin x} = \underline{g_1(x) + g_2(x)}$

Use $g_1(x)$ to guess y_{p_1}

$g_2(x)$ y_{p_2}

$y_p = y_{p_1} + y_{p_2}$

$g_1(x) = x^2 + x - 1$, $P(x) = x^2 + x - 1$ degree 2

$r_0 = 0$

Char eq: $r^2 - 4r + 3 = 0$ has 2 roots $1, 3$

$y_{p_1} = x^m (Ax^2 + Bx + C)$

$m = 0$, $y_{p_1} = Ax^2 + Bx + C$.

$g_2(x) = \sin x$, $P(x) = 1$, $\alpha = 0$, $\beta = 1$

$\alpha + i\beta$ is not a root

$\Rightarrow y_{P_2} = x^m (D \sin x + E \cos x)$

$m = 0$

$y_{P_2} = D \sin x + E \cos x$

$y_p = Ax^2 + Bx + C + D \sin x + E \cos x$

4.6. Variation of Parameters

(Another way to find y_p)

y_p is determined by $g(x)$ and y_c is a concrete way

Consider 2nd order linear equation:

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$$

dividing by $a_2(x)$

$$\Leftrightarrow y'' + P(x) y' + q(x) y = f(x) \quad (1)$$

non-hom component

The associated homogeneous eq:

$$y'' + Py' + qy = 0 \quad (2)$$

$y_c = c_1 y_1 + c_2 y_2$.

Assume that y_1 and y_2 are known then

$y_p = u y_1 + v y_2$

u and v are computed as follows:

$$W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \not\equiv 0$$

$$W_1 = \det\begin{pmatrix} 0 & y_2 \\ f(x) & y_2' \end{pmatrix}, \quad W_2 = \det\begin{pmatrix} y_1 & 0 \\ y_1' & f(x) \end{pmatrix}$$

$$u = \int \frac{W_1}{W}, \quad v = \int \frac{W_2}{W}$$

(How to remember: Replacing a column in W by the nonhomogeneous part)

<u>Explanation</u>: (How u and v are calculated as such)

$y_p = u y_1 + v y_2$

$$\left.\begin{array}{l} y_p' = \cdots \\ y_p'' = \cdots \end{array}\right\} \text{sub in } ① \Rightarrow \text{simplification}$$

(y_1, y_2 are sol to ②)

$\Rightarrow$ equations for u, v, u', v',

$\Rightarrow$ solving for them.

<u>Ex</u>: $y'' + y = \sec(x) = \dfrac{1}{\cos x}$ $g(x)$

Method of undetermined coefficients doesn't work.

Char eq: $r^2 + 1 = 0$, $r = \pm i$, $(\alpha = 0, \beta = 1)$

$y_1 = \cos x$, $y_2 = \sin x$

$$W = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = \cos^2 x - (-\sin^2 x) = 1.$$

$$W_1 = \begin{pmatrix} 0 & \sin x \\ \sec x & \cos x \end{pmatrix} = -\sin x \sec x = -\frac{\sin x}{\cos x}$$

$$W_2 = \begin{pmatrix} \cos x & 0 \\ -\sin x & \sec x \end{pmatrix} = \cos x \sec x = 1.$$

$$u = \int \frac{W_1}{W} = \int -\frac{\sin x}{\cos x} dx = \int \frac{ds}{s} = \ln(s)$$

$$(s = \cos x, \ ds = -\sin x \, dx)$$

$$= \ln|\cos x|$$

$$v = \int \frac{W_2}{W} = \int \frac{1}{1} = x$$

$$y_p = u y_1 + v y_2 = (\ln|\cos x|) \cos x + x \sin x$$